A Graphical Language for Beam Splitters

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Goal: having a formal tool (here a graphical language) to describe the behavior of optical devices made of quantum channels (acting on some data carried by the photon) and beam splitters (used to perform quantum control of the channels).

Example

There is an informal language that is widely used, and has an informal semantics given by the physical behavior.

- Can this language be formalized ?
- Can the semantics be formalized in a compositional way (that is, the semantics of a composite diagram is obtained from the semantics of every element) ?
- Will the language have nice graphical properties ?
- Can we have an equational theory ?

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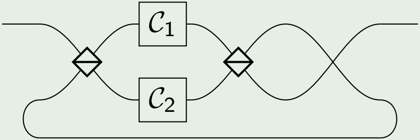

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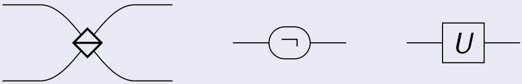
Definition

The devices are described by string diagrams called *PBS-diagrams*, generated by

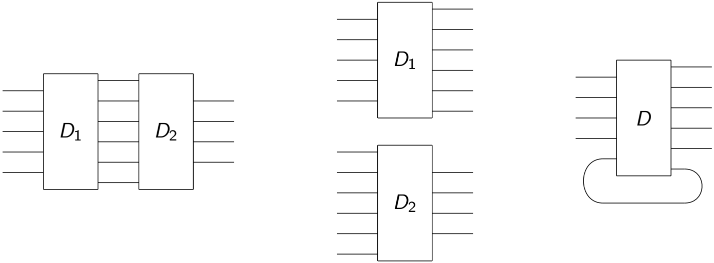


and equipped with a structure of traced PROP.

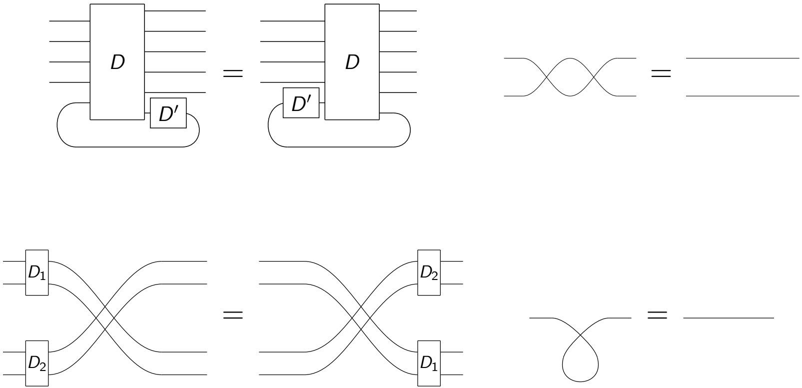
Structure of traced PROP

Two more generators: identity and swap.

Sequential composition, parallel composition and trace.



Structure of traced PROP: only connectivity matters



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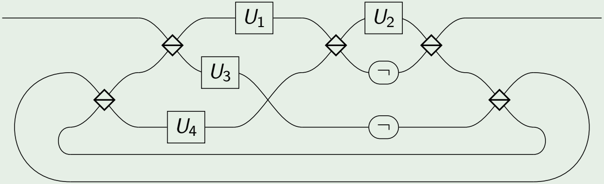

$U_{\rightarrow} = U_2 U_1$
$U_{\uparrow} = U_4 U_3$

Remark

The photon passes at most once in each wire with each polarization $\rightarrow$ or $\uparrow$.

Intuitive "path" semantics

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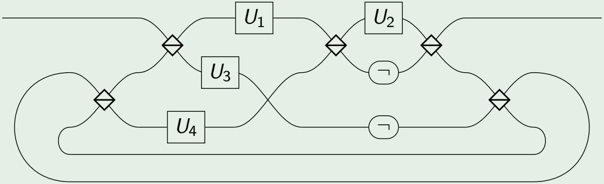

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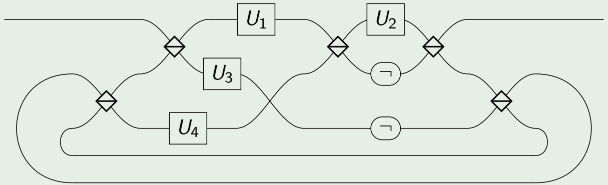

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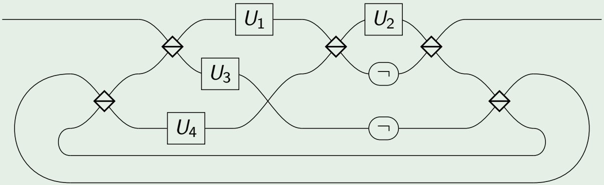

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Formal compositional semantics

The state of the photon is described by a vector in

$$\mathcal{H}_n = \underbrace{\mathbb{C}^{\{\rightarrow,\uparrow\}}}_{\textit{polarization}} \otimes \underbrace{\mathbb{C}^{n}}_{\textit{position}} \otimes \underbrace{\mathbb{C}^{q}}_{\substack{\textit{additional}\\ \textit{data}}}$$

$$= \begin{array}{lcl} |\rightarrow,0,y\rangle & \mapsto & |\rightarrow,0,y\rangle \\ |\rightarrow,1,y\rangle & \mapsto & |\rightarrow,1,y\rangle \\ |\uparrow,0,y\rangle & \mapsto & |\uparrow,1,y\rangle \\ |\uparrow,1,y\rangle & \mapsto & |\uparrow,0,y\rangle \end{array}$$

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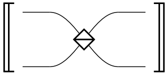

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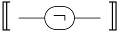

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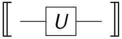

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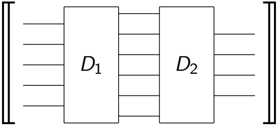
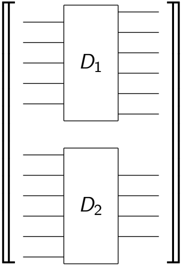
Formal compositional semantics

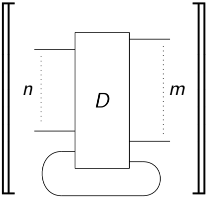


$$= \llbracket D_2 \rrbracket \circ \llbracket D_1 \rrbracket$$



$$= \llbracket D_1 \rrbracket \oplus \llbracket D_2 \rrbracket$$

Formal compositional semantics



$$= \sum_{k\in\mathbb{N}} \pi_1 \circ (\llbracket D \rrbracket \circ \pi_0)^k \circ \llbracket D \rrbracket \circ \iota$$

where $\iota : \mathcal{H}_n \to \mathcal{H}_{n+1} :: |c,p,y\rangle \mapsto |c,p,y\rangle$

$$\pi_0 : \mathcal{H}_{m+1} \to \mathcal{H}_{n+1} :: |c,p,y\rangle \mapsto \begin{cases} 0 & \text{if } p < m \\ |c,n,y\rangle & \text{if } p = m \end{cases}$$

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Formal compositional semantics

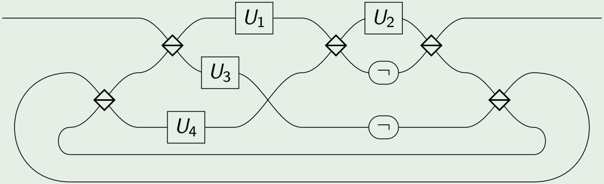
Theorem

For any diagram D, we have

$$[\![D]\!] = |c, p, y\rangle \mapsto |s(c, p)\rangle \otimes U_{c,p} |y\rangle$$

where $U_{c,p}$ is the product of the unitary matrices encountered, and $s(c, p)$ is the couple (c', p') of the final polarization and position.

Example



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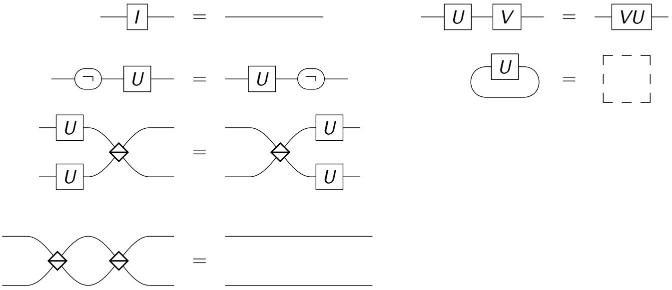
$$\begin{aligned}[\![D]\!] &= |\rightarrow, 0, y\rangle \mapsto |\rightarrow, 0\rangle \otimes U_\rightarrow |y\rangle \\ &= |\uparrow, 0, y\rangle \mapsto |\uparrow, 0\rangle \otimes U_\uparrow |y\rangle\end{aligned}$$

Set of transformation rules



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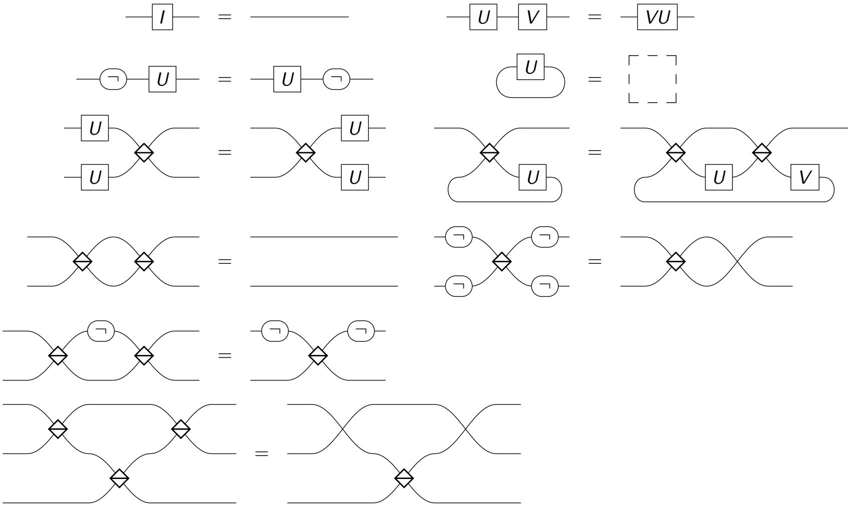

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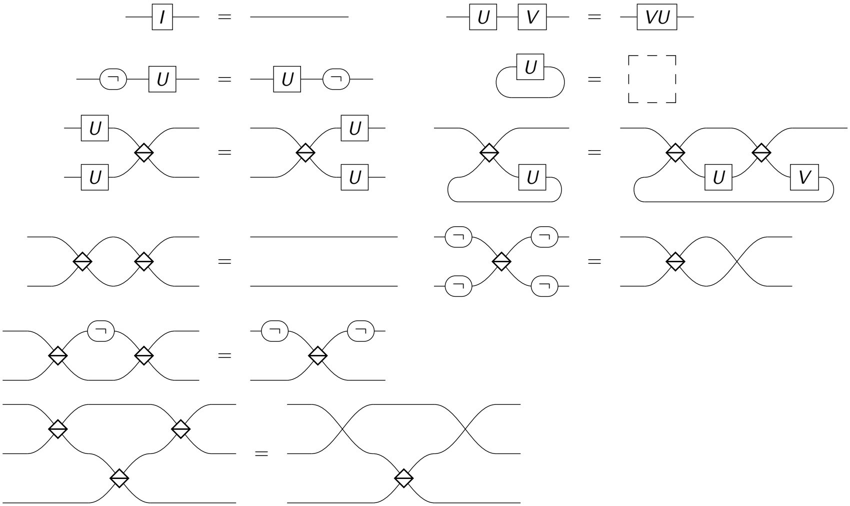
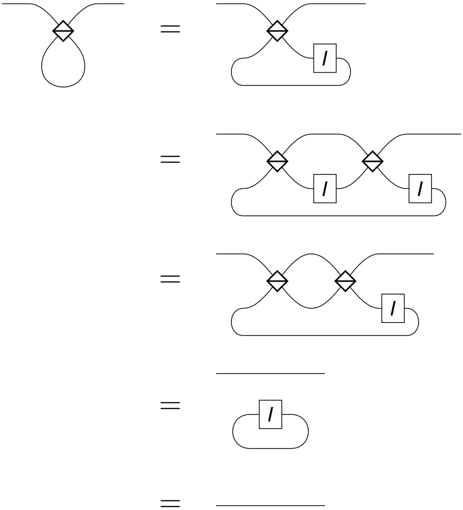


Soundness: $\forall D_1, D_2, PBS \vdash D_1 = D_2 \Rightarrow [\![D_1]\!] = [\![D_2]\!]$.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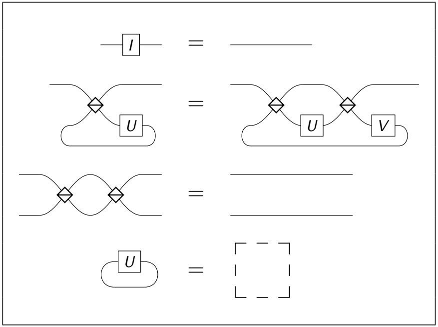
I
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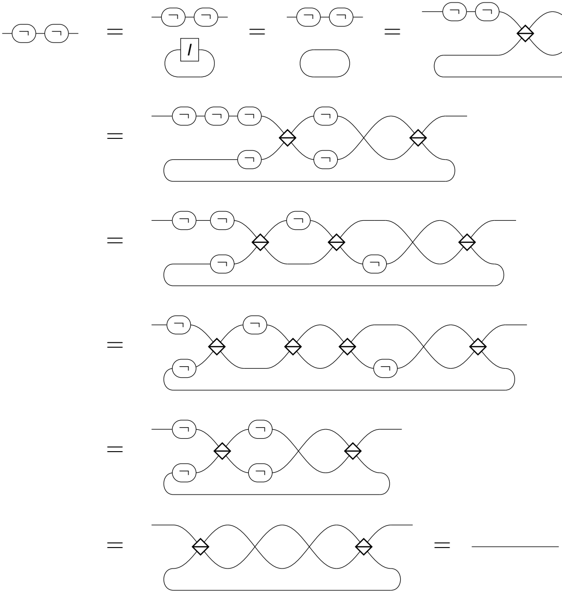

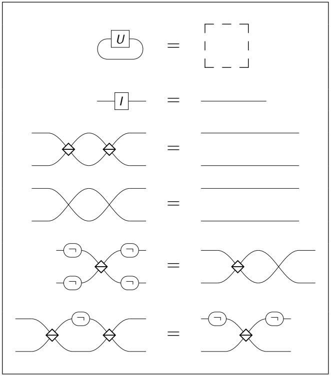

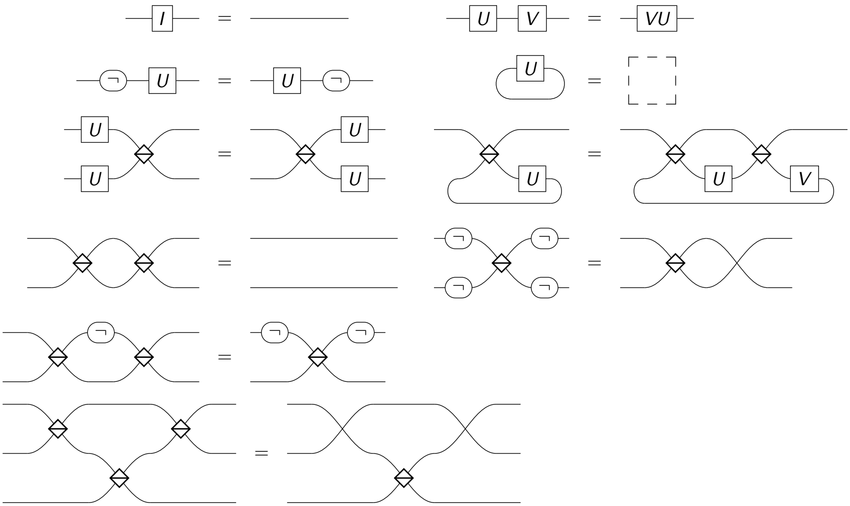

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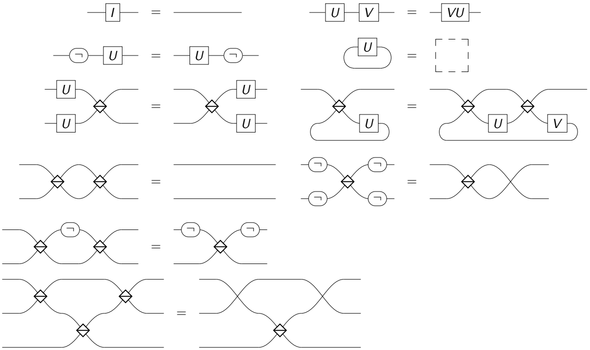

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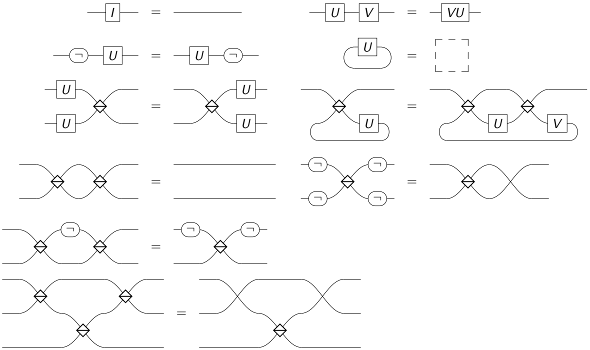

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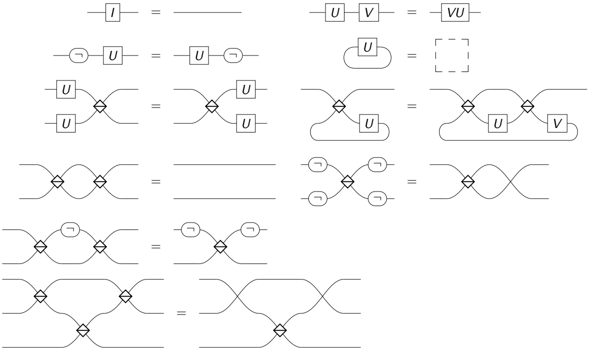
Theorem (completeness)

$\forall D_1, D_2, [\![D_1]\!] = [\![D_2]\!] \Rightarrow PBS \vdash D_1 = D_2.$

Theorem (Minimality)

None of the equations of PBS is a consequence of the others.



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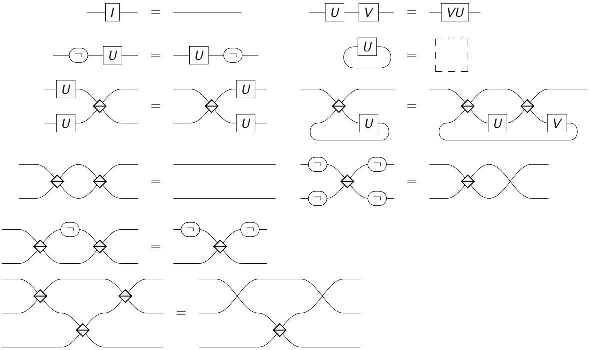

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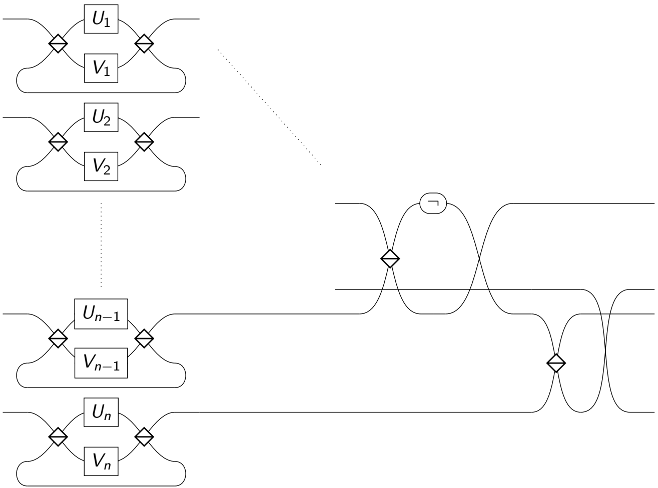
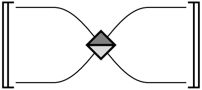
Normal form



Non-polarizing beam splitter



$$= |c, p, y\rangle \mapsto |c\rangle \otimes (H\,|p\rangle) \otimes |y\rangle$$

$$\text{where } H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Theorem

For any diagram $D : n \to n$, $[\![D]\!]$ is unitary.

Theorem (Universality)

For any unitary map $U\colon \mathcal{H}_n \to \mathcal{H}_n$, there exists a diagram D such that $[\![D]\!] = U$.

Future work

- Extending the axiomatization to non-polarizing beam splitters.
- Adding non-unitary channels.
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Thanks for listening !