Hypegraph-Based Contextuality

Mladen Pavičić

Center of Excellence for Advanced Materials and Sensors (CEMS),
Research Unit *Photonics and Quantum Optics*,
Institute Ruđer Bošković (IRB), Zagreb, Croatia.

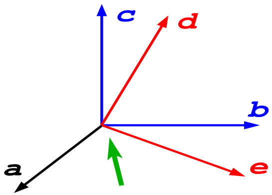
Journées Informatique Quantique 2019,

Nov 28 & 29, 2019 - Besançon, France.

Besançon2019

McKay-Megill-Pavičić hypergraph (MMPH) strings

vectors



orthogonalities

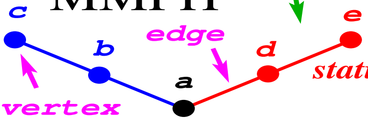
system of nonlinear equations

$$\mathbf{a}\cdot\mathbf{b} = a_x b_x + a_y b_y + a_z b_z = 0$$
$$\mathbf{a}\cdot\mathbf{c} = a_x c_x + a_y c_y + a_z c_z = 0$$
$$\mathbf{b}\cdot\mathbf{c} = b_x c_x + b_y c_y + b_z c_z = 0$$
$$\mathbf{a}\cdot\mathbf{d} = a_x d_x + a_y d_y + a_z d_z = 0$$
$$\mathbf{a}\cdot\mathbf{e} = a_x e_x + a_y e_y + a_z e_z = 0$$
$$\mathbf{d}\cdot\mathbf{e} = d_x e_x + d_y e_y + d_z e_z = 0$$

coordinatization

exponential complexity

MMPH



statistically polynomial complexity

MMPH string: **cba,ade.**

Vertex notation; Edge; Hypergraph

hypergraph = pair v-e;

v = a set of elements called vertices;

e = a set of non-empty subsets of e called edges;

edge = a set of vertices *related* to each other — e.g., *orthogonal* to each other.

Each vertex is denoted by one of the following characters:

1 2 ... 9 A B ... Z a b ... z ! " # $ % & ' ( ) * - / : ; < = > ? @ [\ ] ^ _ ` { | } ~, +1, +2, ... +Z, +a, +b, ...+~, ++1, ++2, ... ++Z, ++a, ++b, ...

McKay-Megill-Pavičić hypergraph (MMPH)

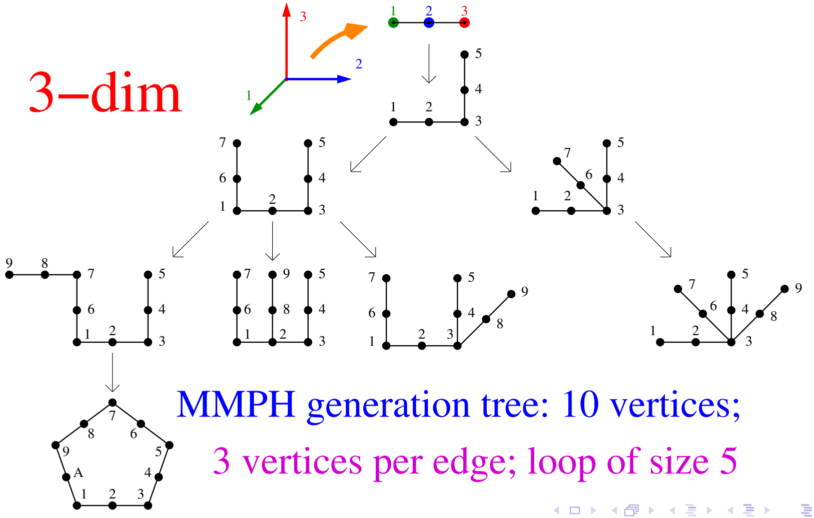
An MMPH is an *n*-dim hypergraph in which

(i) Every vertex belongs to at least one edge;

(ii) Every edge contains at least 2 vertices;

(iii) Edges that intersect each other in $m-2$ vertices contain at least m vertices, $2 \leq m \leq n$.

Isomorphism-free MMPH generation: M Pavičić, J-P Merlet, B D McKay & N D Megill, *J. Phys. A*, **38**, 1577 (2005)



MMPH generation tree: 10 vertices;

3 vertices per edge; loop of size 5

Enter MMPH non-contextuality (N-C) & contextuality

n-dim MMPH $\mathrm{non\text{-}binary\ contextual\ set}$, $n \geq 3$, is a hypergraph whose each edge contains at least two and at most n vertices to which it is impossible to assign 1s and 0s in such a way that

No two vertices within any of its edges are both assigned the value 1;

In any of its edges, not all of the vertices are assigned the value 0.

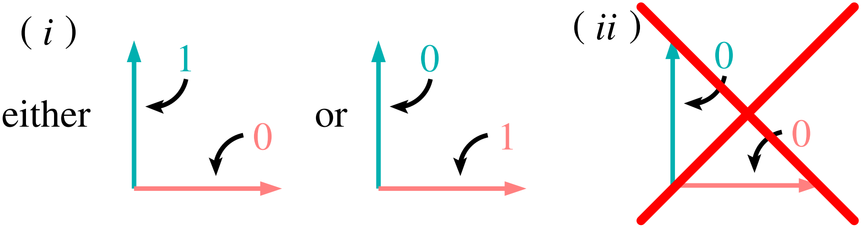
****

An MMPH set to which it is possible to assign 1s and 0s so as to satisfy the above two conditions is a $\mathrm{N\text{-}C}$ MMPH $\mathrm{binary\ set}$.

An MMPH non-binary set with edges of mixed sizes to which vertices are added so as to make all edges of equal size each containing n vertices is called *filled* MMPH set.

MMPH non-binary set conditions visualised

MMPH non–binary set violates the following conditions:



Measuring MMPH non-contextuality vs. contextuality

quantum hypergraph index (HI_q) = sum of probabilities of getting detector clicks for all considered vertices

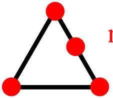
classical hypergraph index (HI_c) = maximal number of 1s assigned to vertices so as to satisfy the two conditions from the previous slide.

Non-contextual inequality - contextual distinguisher

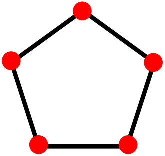
Contextual, non-binary sets: $HI_q > HI_c$

Non-contextual, binary sets: $HI_q \leq HI_c$

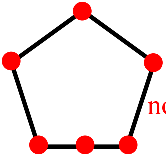
MMPH coordinatization and contextuality



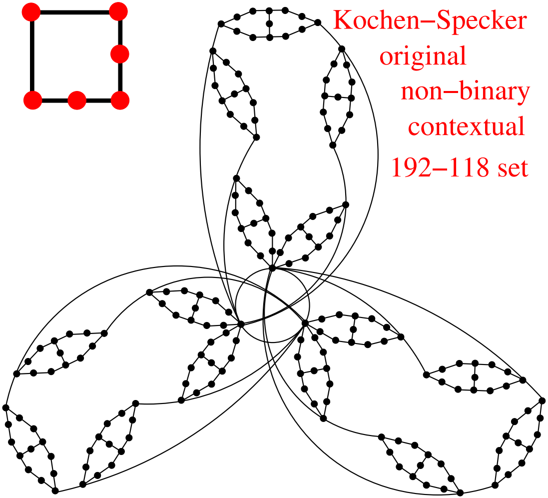
no coordinatization



non–binary
contextual
pentagon

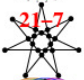


binary
non–contextual
pentagon

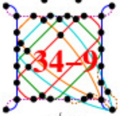
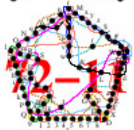


Kochen–Specker
original
non–binary
contextual
192–118 set

MMPH KS masters: Coordinatization inherited

dim	Master Size	Vector List	List Origin	Smallest Hypergraph	Vector Components
4D	24-24	[25,42,43]	symmetry, geometry		$\{0, \pm 1\}$
4D	60-105	[28,37]	Pauli operators		$\{0, \pm 1, \pm i\}$
4D	60-75	[27,30,37,41]	regular polytope 600-cell		$\{0, \pm(\sqrt{5}-1)/2, \pm 1, \pm(\sqrt{5}+1)/2, 2\}$
4D	148-265	[36,37]	Witting polytope		$\{0, \pm i, \pm 1, \pm\omega, \pm\omega^2, \pm i\omega^{1/\sqrt{3}}, \pm i\omega^{2/\sqrt{3}}\}$
6D	21-7	[19]	symmetry		$\{0, 1, \omega, \omega^2\}$
6D	236-1216	Aravind & Waegell 2016, [37]	hypercube →hexaract Schäfli $\{4, 3^4\}$		$\{0, \pm 1/2, \pm 1/\sqrt{3}, \pm 1/\sqrt{2}, 1\}$

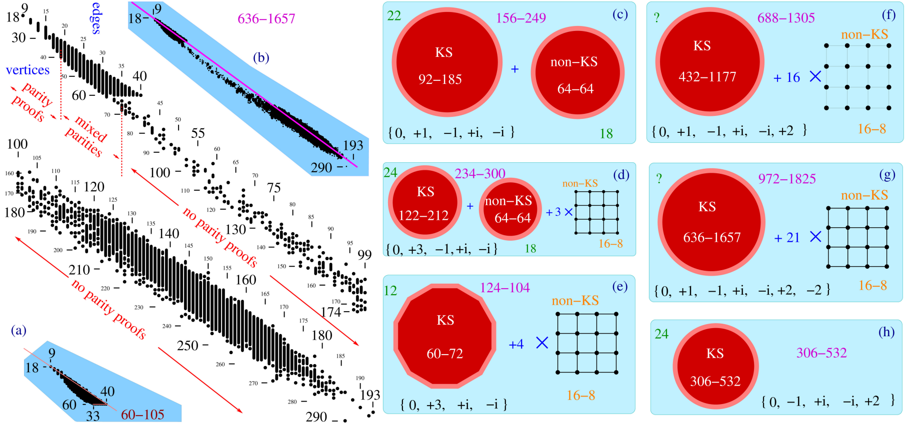
M. Pavičić and N.D. Megill, Vector Generation of Quantum Contextual Sets in Even Dimensional Hilbert Spaces, *Entropy* **20**(12), 928 (2018)

8D	36-9	[37]	symmetry		$\{0,\pm1\}$
8D	120-2025	[35,37]	Lie algebra E8		as given in [35]
16D	80-265	[37,44,45]	Qubit states		$\{0,\pm1\}$
32D	160-661	[37,46]	Qubit states		$\{0,\pm1\}$

MMPH KS masters together with their coordinatization created from simple vector components

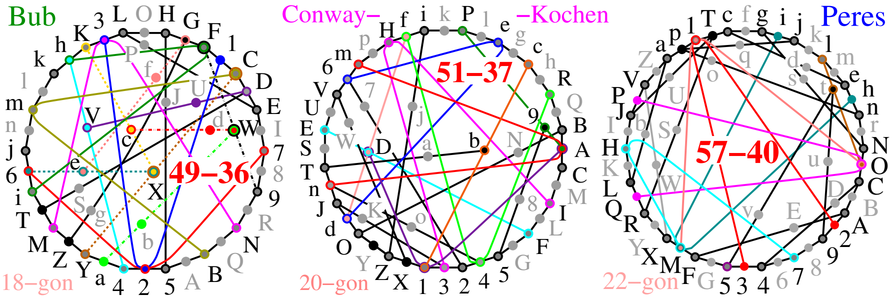
dim	Vector Components	Component-Master Size	Nº of KS Criticals in Master	Smallest Hypergraph	Contains List-Masters
4D	$\{0,\pm1\}$ or $\{0,\pm i\}$ or $\{0,\pm(\sqrt{5}-1)/2\}$ or ...	40-32	6		24-24
4D	$\{0,\pm1,\pm i\}$	156-249	7.7×10^6		24-24, 60-105
4D	$\{0,\pm(\sqrt{5}-1)/2,\pm1,$ $\pm(\sqrt{5}+1)/2,2\}$	2316-3052	1.5×10^9		24-24, 60-75
4D	$\{0,\pm1,\pm i,\pm\omega,\pm\omega^2\}$	400-1012	8×10^6		24-24, 60-105 148-265
6D	$\{0,\pm1,\omega,\omega^2\}$	11808-314446	3×10^7		21-7, 236-1216
8D	$\{0,\pm1\}$	3280-1361376	7×10^6		36-9, 120-2025
16D	$\{0,\pm1\}$	computationally too demanding	4×10^6	?–9 [33].	80-265
32D	$\{0,\pm1\}$	computationally too demanding	2.5×10^5	?–9 [33].	160-661

Examples: M. Pavičić, M. Waegel, N. Megill, and P.K. Aravind, Automated generation of Kochen-Specker sets, *Scientific Reports*, **9**, 6765 (2019).



M. Pavičić, Arbitrarily exhaustive hypergraph generation of 4-, 6-, 8-, 16-, and 32-dimensional quantum contextual sets, *Physical Review A*, **95**, 062121–1-25 (2017).

Gray vertices are usually dropped in the literature



33–36 31–37 33–40

Literature names

In the literature all vertices that appear in only one edge are dropped, but ...

Peres wrote "It can be shown that if a single ray is deleted from the set of 33, the contradiction disappears." [A. Peres, *J. Phys. A*, **24**, L175 (1991)]

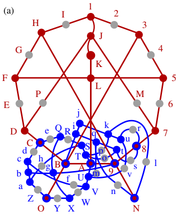
"In the original proof of Kochen and Specker the number of elements is 117. The present record, due to Kochen and Conway, is 31 vectors." [I. Pitowsky, *J. Math. Phys.*, **39**, 218 (1998)

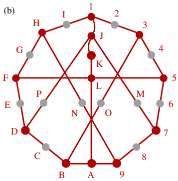
Similar statements throughout the literature.

But none of them: Bub's 33-36, Conway-Kochen's 31-37, Peres' 33-40, and Kochen-Specker's 117-118, is actually critical, i.e., if a single vertex/ray/vector or edge were deleted, the "contradiction" would not disappear.

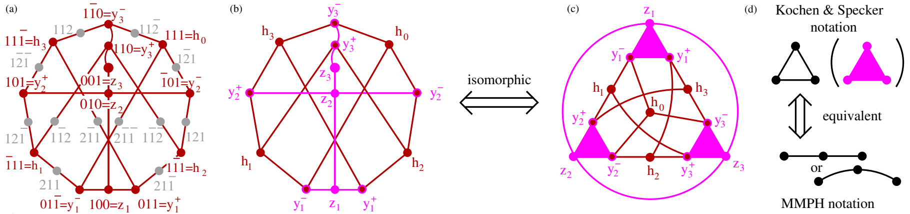
S. Yu and C.H. Oh, State-Independent Proof of Kochen-Specker Theorem with 13 Rays, *Phys. Rev. Lett.* **108**, 030402 (2012).

(a) Peres' 57-40 critical MMPH non- binary KS set containing Yu-Oh *filled* 25-16 MMPH binary non-KS set.





(b) Removal of gray vertices from 25-16 yields MMPH non-binary Yu-Oh 13-16 set. See (b,c) below.



Yu & Oh did not prove the Kochen-Specker theorem but they introduced a kind of operator-based contextuality and its non-contextual inequality

Yu & Oh picked 13 vertices out of 25 to construct an expression of state/vector defined 3x3 operators that eventually reduces to a multiple of a unit operator.

I. Bengtsson, K. Blanchfield, and A. Cabello, A Kochen-Specker Inequality from a SIC, *Phys. Lett. A*, **376**, 374 (2012) and

Z.P. Xu, J.L. Chen, and H.Y. Su, State-independent contextuality sets for a qutrit. *Phys. Lett. A*, **379**, 1868 (2015)

make use of projectors whose expressions also reduce to a multiple of a unit operator.

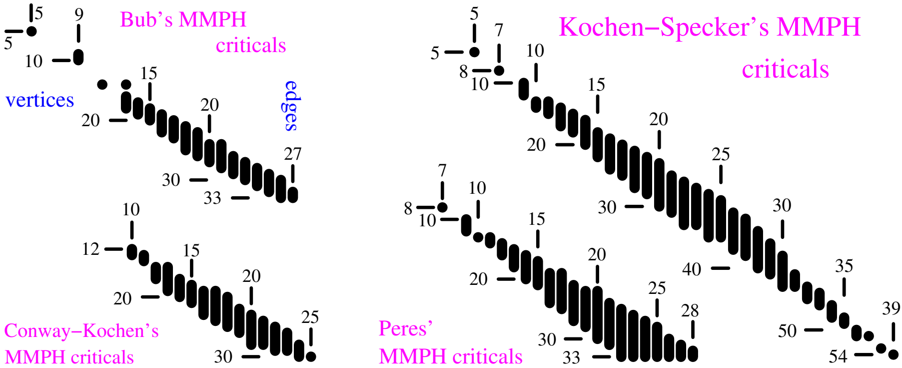
This takes us to M. Pavičić, Hypergraph contextuality, *Entropy*, **21**(11), 1107 (2019)

Bub's 33-36, Conway-Kochen's 31-37, Peres' 33-40, and Kochen-Specker's 117-118 are not critical

$\Rightarrow$ we take them (or other MMPHs smaller then Bub's 49-36, Conway-Kochen's 51-37, Peres' 57-40, and Kochen-Specker's 192-118) as our master MMPH sets.

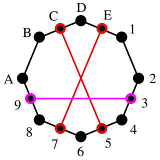
Via our algorithms and programs we obtain smaller sub-MMPHs

Distribution of sub-MMPHs

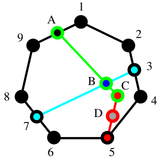


Peres' non-binary non-KS 33-40 MMPH master set: `123,345,47,79,92A, AC,C4,AF,5F,HJ,HL,H7M,NCO,OPQ,QRL,RT,TJ,JPV,VX,XR,Va,La,ce, cT1,cg,FXM,Mhi,ijg,jl,le,ehn,np,pj,nN,gN,t9,tlO,t5,ap1,1MO.`

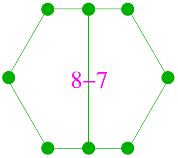
Samples of sub-MMPHs



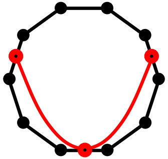
bub–14–11



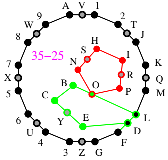
conway–kochen–13–10



kochen–specker–8–7



peres–13–11



kochen–specker–35–25

bub-14-11 and peres-13-11 are some of the very few 3-dim MMPHs with a parity proof.

Operator-based inequalities for Yu-Oh's 13-16 set

$$\hat{L}_{13} = \sum_{i}^{13} \hat{A}_i - \frac{1}{4} \sum_{i}^{13} \sum_{j}^{13} \Gamma_{ij} \hat{A}_i \hat{A}_j = \frac{25}{3} I = 8.\dot{3} I, \quad (1)$$

where $\Gamma_{ij} = 1$ whenever corresponding vectors i, j are orthogonal to each other and $\Gamma_{ij} = 0$ when they are not; also $\Gamma_{ii} = 0$.

Corresponding expression for 13 classical variables with predetermined values -1 and 1:

$$C_{13} = \sum_{i}^{13} a_i - \frac{1}{4} \sum_{i}^{13} \sum_{j}^{13} \Gamma_{ij} a_i a_j \leq 8 \quad (2)$$

$$\mathrm{Yu{-}Oh\ inequality:} \qquad 8.\dot{3} = \langle \hat{L} \rangle > Max[C] = 8 \quad (3)$$

We also calculated Yu-Oh-like inequalities for 50 sets different from Yu-Oh's 13–16 one. None of them satisfied the inequality.

Experimental MMPH calibration

Measurements at MMPH triplets gates, are carried out with the $1/3$ probability of detection at each out-port.

Measurements at MMPH doublet gates, are *calibrated* so as to have the $1/2$ probability of getting a click at either of the two considered ports, while ignoring the third one, meaning that the inputs to doublet gates should be scaled up with respect to the full triplet ones by $3/2$ to assure an equal distribution of outcomes at each port,.

When a vertex shares a mixture of triplet and doublet edges the probability of detection is p, where $1/3 \leq p \leq 1/2$.

We call detections at all ports, notwithstanding whether we include them in our final statistics or not, *uncalibrated* detections—they simply have the $1/3$ probability of detection at every port.

Non-contextual inequalities for MMPHs

Calibrated MMPH non-contextual inequality for 14-11:
$HI_q[\text{14-11}] = 4\times 1/3+10\times(1/2+1/3)/2 = 11/2 = 5.5\dot{3} > HI_c[\text{14-11}] = 5.$

Uncalibrated MMPH Non-contextual inequality for 14-11:
$HI_q[\text{14-11}] = 14/3 = 4.\dot{6} < HI_c[\text{14-11}] = 5.$

For some other MMPHs we have:

Yu-Oh's 13-16 (calibrated): $HI_q[\text{13-16}]=17/3=5.\dot{6} > HI_c[\text{13-16}]=4;$

Yu-Oh's 13-16 (uncalibrated): $HI_{q-unc}[\text{13-16}]=13/3=4.\dot{6} > HI_c[\text{13-16}]=4.$

$HI_q[\text{13-10}]=4.9\dot{4} > HI_{q-unc}[\text{13-10}]=4.\dot{3} > HI_c[\text{13-10}]=4$

$HI_q[\text{35-25}]=13.75\dot{4} > HI_c[\text{35-25}]=12 > HI_{q-unc}[\text{35-25}]=11.\dot{6}$

bub $HI_{q-unc}[\text{33-36}] = 11 > HI_c[\text{33-36}] = 10$
conway-kochen $HI_{q-unc}[\text{31-37}] = 10.\dot{3} > HI_c[\text{31-37}] = 8$
peres $HI_{q-unc}[\text{33-40}] = 11 > HI_c[\text{33-40}] = 6$

Quantum Computation Magic

QUANTUM COMPUTING

Powered by magic

What gives quantum computers that extra oomph over their classical digital counterparts? An intrinsic, measurable aspect of quantum mechanics called contextuality, it now emerges. SEE ARTICLE P.351

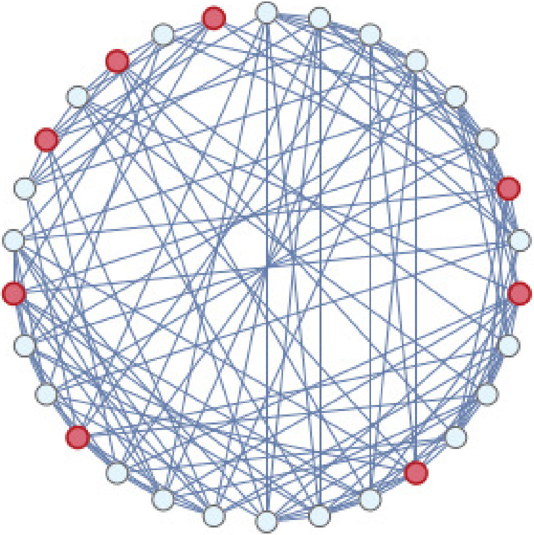
ARTICLE 19 JUNE 2014 | VOL 510 | NATURE | 351

doi:10.1038/nature13460

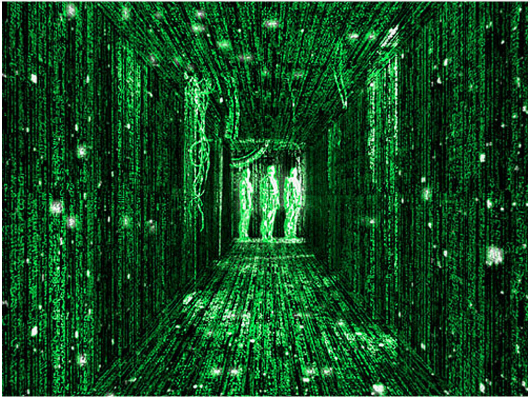
Contextuality supplies the 'magic' for quantum computation

Mark Howard[1,2], Joel Wallman[2], Victor Veitch[2,3] & Joseph Emerson[2]

Maximum independent set of 2-qubit stabilizer states



Thanks for your attention 😎



http://cems.irb.hr/en/research-units/photonics-and-quantum-optics/
http://www.irb.hr/users/mpavicic/

Europska unija Zajedno do fondova EU
Ministarstvo znanosti i obrazovanja
Ministarstvo regionalnog razvoja i fondova Europske unije
Operativni program KONKURENTNOST I KOHEZIJA
EUROPSKI STRUKTURNI I INVESTICIJSKI FONDOVI
safu